STABILITY OF A GRAVITATING FLUID LAYER OF UNIFORM THICKNESS IN THE PRESENCE OF CORIOLIS FORCE AND A MAGNETIC FIELD

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ABSTRACT. The stability of a gravitating fluid layer of finite thickness in the presence of a magnetic field and a Coriolis force is studied by the use of the normal mode method. The previous results obtained by Oganesyan are derived from the general result of the present case. It is found that suitably high values of angular velocity of rotation can stabilize the fluid layer completely for the symmetric perturbations in the absence of the magnetic field. But the presence of the magnetic field has a destabilizing influence.

INTRODUCTION

Safranov (1960) has pointed out the importance of the study of gravitational and magneto-grativational stability of fluid layers of finite thickness in the astronomical context. Oganesyan (1961) has recently studied the gravitational and magnetogravitational stability of a fluid layer of infinite extension but of finite thickness. He takes the conductivity of the fluid as infinite. In the present paper we have studied the corresponding problem in the presence of a Coriolis force arising due to the rotation of the fluid layer uniformly about an axis which is normal to the plane of the fluid surface in equilibrium.

Our conclusion is that rotation affects, in a very important way, the stability criterion in the absence of a magnetic field. We find that for the angular velocity of rotation and for the density appropriate for our galaxy, rotation can stabilize fully the fluid layer.

But the magnetic field is able to suppress this effect of rotation and it appears that stability criterion is unaffected by rotation in this case.

We employ the normal mode method in contrast with the energy method used by Oganesyan (1961). Oganesyan's all results follow as particular cases of our general result.

BASIC EQUATIONS

The basic equations in the steady case holding in the infinitely conducting fluid layer are the following.

$$\rho\left\{\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}+2\,\underline{\Omega}\times\boldsymbol{V}+\underline{\Omega}\times(\underline{\Omega}\times\boldsymbol{r})\right\}=-\nabla p+\mu\,\mathrm{curl}\,\boldsymbol{H}\times\boldsymbol{H}-\rho\nabla U, \quad \ldots (1)$$

$$\operatorname{div} \boldsymbol{V} = 0, \qquad \dots \quad (2)$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}(\boldsymbol{V} \times \boldsymbol{H}), \qquad \dots \quad (3)$$

$$\operatorname{div} \boldsymbol{H} = 0, \qquad \dots \quad (4)$$

$$\nabla^2 U = -4\pi G\rho. \qquad \dots \quad (5)$$

In the above we have used the M.K.S. system of units. The symbols Ω, U and G represents the angular velocity of rotation of the fluid layer, the gravitational potential and the universal gravitational constant. The other symbols have the usual meanings.

We shall take Ω to be along the z-axis. The fluid layer in steady state is confined within the planes $z = \pm h$, $2h$ being the thickness of the layer.

The equations holding in the surrounding non-conducting fluid, which is also assumed to be rotating in the steady state with angular velocity Ω, are

$$\rho_0 \left\{ \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + 2\Omega \times \boldsymbol{V} + \Omega \times (\Omega \times \boldsymbol{r}) \right\} = -\nabla P + \rho_0 \nabla V, \qquad (6)$$

$$\operatorname{div} \boldsymbol{V} = 0, \qquad \dots \quad (7)$$

$$\operatorname{curl} \boldsymbol{H} = 0, \qquad \dots \quad (8)$$

$$\nabla^2 V = -4\pi G\rho_0 \qquad \dots \quad (9)$$

where ρ_0 is the density of the non-conducting fluid and V is the gravitational potential there.

The magnetic field in the steady (equilibrium) state is taken to be uniform and along x-axis. It is denoted by $\boldsymbol{H}_0$.

For small perturbations about the equilibrium state we have, after linearization, the following equations.

$$\rho\left(\frac{\partial \widetilde{\boldsymbol{V}}}{\partial t} + 2\Omega \times \widetilde{\boldsymbol{V}} \right) = -\nabla \tilde{p} + \mu \operatorname{curl} \widetilde{\boldsymbol{H}} \times \boldsymbol{H}_0 + \rho \nabla \widetilde{\boldsymbol{U}} \qquad \dots \quad (10)$$

$$\operatorname{div} \widetilde{\boldsymbol{V}} = 0 \qquad \dots \quad (11)$$

$$\frac{\partial \widetilde{\boldsymbol{H}}}{\partial t} = \operatorname{curl} (\widetilde{\boldsymbol{V}} \times \boldsymbol{H}_0) \qquad \dots \quad (12)$$

$$\operatorname{div} \widetilde{\boldsymbol{H}} = 0, \qquad \dots \quad (13)$$

$$\nabla^2 \widetilde{U} = 0, \qquad \dots \quad (14)$$

holding in the conducting fluid. Similar equations for small perturbations are obtained from equations 6– 9 which hold for the non-conducting fluid.

Solutions of the Perturbation Equations

If q is any physical quantity the perturbation in it is denoted by $\tilde{q}$. We shall assume, in the further discussion, $\tilde{q}$ to be of the form

$$\tilde{q} = \tilde{q}(z)k^{i(\sigma t+kx+ly)} \qquad \dots \quad (15)$$

where σ is the frequency of oscillations and k and l are wave numbers in x and y directions.

Solutions in the Conducting Fluid Layer

The equation (12) in view of (11) gives

$$\tilde{\boldsymbol{H}} = \frac{kH}{\sigma} \tilde{\boldsymbol{v}}, \qquad \dots \quad (16)$$

where

$$|\boldsymbol{H}_0| = H.$$

Taking the curl of (10), and using (11), (15) and the constancy of $\boldsymbol{H}_0$ and Ω we have

$$A\boldsymbol{w} = -2\Omega \quad \partial z \;, \qquad \dots \quad (17)$$

where

$$\bar{\boldsymbol{w}} = \text{curl}\,\bar{\boldsymbol{V}} \qquad \dots \quad (18)$$

and

$$A = i\sigma - \frac{\eta\mu k^{2}}{\rho\sigma} \qquad \dots \quad (19)$$

Using (11), we have from the z-component of (17)

$$v_y = \frac{\tilde{v}_x(Al - 2\Omega k)}{2\Omega l + Ak} \qquad \dots \quad (20)$$

In view of (11) and (20), we have from (17),

$$D^2\tilde{v}_x = m^2\tilde{v}_x, \qquad \dots \quad (21)$$

where

$$m^2 = \frac{A^2(k^2+l^2)}{A^2+4\Omega^2} \qquad \dots \quad (22)$$

We shall consider perturbations symmetrical about $z = 0$ plane. Considering the distribution of $\hat{v}_z$ about $z = 0$, it is clear that $\hat{v}_z$ is an odd functions of z. Using the above result we have from (11), (20) and (21)

$$\left.\begin{aligned} \hat{v}_x &= C \cosh mz \\ \hat{v}_y &= \frac{\{Al - 2\Omega k\}}{\{2\Omega l + Ak\}} C \cosh mz \\ \hat{v}_z &= \frac{i\{A(k^2+l^2)\}C \sinh mz}{m\{2\Omega l + Ak\}} \end{aligned}\right\} \qquad \ldots (23)$$

when C is an arbitrary constant.

From the symmetry of the configuration we have the condition $\frac{\partial \hat{U}}{\partial z} = 0$ at $z = 0$. In view of this condition we have from (14)

$$\hat{U} = G' \cosh (k^2+l^2)^{\frac{1}{2}} z. \qquad \ldots (24)$$

Similarly, we can easily find

$$\hat{\Gamma} = G'' e^{-\sqrt{k^2+l^2}\, z} \qquad \ldots (25)$$

The velocity components for the non-conducting fluid are

$$\left.\begin{aligned} \hat{v}_x &= C_1 e^{-m'z} \\ \hat{v}_y &= \frac{(i\sigma l - 2\Omega k)}{2\Omega l + i\sigma k} \hat{v}_x, \\ \hat{v}_z &= -\frac{\sigma(k^2+l^2)C_1 e^{-m'z}}{m'(2\Omega l + i\sigma k)} \end{aligned}\right\} \qquad \ldots (26)$$

where C_1 is an arbitrary constant and

$$m'^2 = \frac{\sigma^2(k^2+l^2)}{\sigma^2 - 4\Omega^2}. \qquad \ldots (27)$$

From the equations for the perturbations it is clear that $\widetilde{H}$ is given by

$$\widetilde{H} = \operatorname{grad} \widetilde{\phi}, \qquad \nabla^2 \widetilde{\phi} = 0, \qquad \ldots (28)$$

in the non-conducting fluid. In view of (15) we have

$$\tilde{\phi} = L\, e^{-\sqrt{k^2+l^2}\, z} \qquad \ldots (29)$$

Boundary Conditions and the Dispersion Relations

If $$z = h + \delta z e^{i(\sigma t+kx+ly)} \qquad \ldots \quad (30)$$

is the equation of the upper interface between the conducting and the nonconducting fluids, the perturbation $\widetilde{\boldsymbol{n}}$ in the unit normal is given by

$$\widetilde{\boldsymbol{n}} = (ik\delta z,\ il\delta z,\ 0)e^{i(\sigma t+kx+ly)} \qquad \ldots \quad (31)$$

The gravitational potential satisfies the following boundary conditions at the interface of the two fluids :

(i) Continuity of the gravitational potential;

(ii) Continuity of the normal component of the gradient of the gravitational potentials.

The perturbations in the gravitational potentials, satisfying the above boundary conditions are

$$\left.\begin{aligned} \hat{\Gamma} &= \frac{4\pi i G(\rho-\rho_0)\hat{v}_z(h)\cosh(k^2+l^2)^{\frac{1}{2}}h\, e^{-\sqrt{k^2+l^2}\,z}}{\sigma(k^2+l^2)^{\frac{1}{2}}} \\ \hat{\Gamma} &= \frac{4\pi i G(\rho-\rho_0)\hat{v}_z(h)\cosh(k^2+l^2)^{\frac{1}{2}}z}{\sigma(k^2+l^2)^{\frac{1}{2}}\, e^{\sqrt{k^2+l^2}\,h}} \end{aligned}\right\} \qquad (32)$$

The other boundary conditions required may be written as follows (cf. Kruskal and Schwarzchild, 1954).

$$u = \boldsymbol{n}\cdot\boldsymbol{V}, \qquad \boldsymbol{n}\times[\boldsymbol{B}] = \mu_0\boldsymbol{j}^* \qquad \ldots \quad (33, 34)$$

$$\boldsymbol{n}\cdot[\boldsymbol{B}] = 0, \qquad \boldsymbol{j}^*\times\boldsymbol{B}-\boldsymbol{n}[p] = 0. \qquad \ldots \quad (35,36)$$

In the above $\boldsymbol{n}$ is the unit normal vector (directed in the conducting fluid) of surface. p and $\boldsymbol{V}$ are the pressure and velocity on the interface. The brackets denote the jump in the enclosed quantities upon crossing the interface from the non-conducting to the conducting fluid. $\boldsymbol{B}$ denotes the arithmetic mean of the magnetic induction just on each side of the interface.

The condition (35), in view of (33), implies $\hat{H}_z = 0$, in the nonconducting fluid. Hence (28) and (29) show that the perturbations in the magnetic field vanish in the non-conducting fluid.

The boundary conditions satisfied by the perturbations are obtained from 33-36 in the usual manner by linearization. Eliminating $\hat{j}^*$ with the help of

(34), we find that the x- and y-components of (36) are identically satisfied. The z-component finally gives the dispersion relation

$$\left[\frac{4\pi G(\rho-\rho_0)^2}{\sigma(k^2+l^2)^{\frac{1}{2}}\{1+\tanh (k^2+l^2)^{\frac{1}{2}}h\}}+\frac{\rho_0(\sigma^2-4\Omega^2)^{\frac{1}{2}}}{(k^2+l^2)^{\frac{1}{2}}}+\frac{4\pi G\rho h(\rho_0-\rho)}{\sigma}\right]$$

$$\times\frac{\sqrt{k^2+l^2}\sqrt{A^2+4\Omega^2}}{2\Omega l+Ak}\times\tanh\left\{\frac{A\sqrt{k^2+l^2}}{\sqrt{A^2+4\Omega^2}}\right\}h+\left\{\frac{\rho}{k}\left[\sigma+\frac{2i\Omega(Ab-2\Omega k)}{2\Omega l+Ak}\right]\right.$$

$$\left.-\frac{\mu H^2 k}{\sigma}\right\}=0. \qquad \ldots \quad (37)$$

DISCUSSION

We shall first discuss the dispersion relation in special cases before considering it in the general case.

Case I. $\qquad \Omega=0,\ \rho_0=0,\ H=0.$

In this case, in the absence of rotation, magnetic field and outside matter, the dispersion relation (37) reduces to

$$4\pi G\rho h\left[1-\frac{1}{h\sqrt{k^2+l^2}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}}\right]$$

$$\times\sqrt{k^2+l^2}\tanh\sqrt{k^2+l^2}\,h=\sigma^2. \qquad \ldots \quad (38)$$

This is the result obtained by Oganesyan (1961). He finds that the layer is unstable for all waves for which $\sqrt{k^2+l^2}\,h<g_0=0.64$ and it is stable when $\sqrt{k^2+l^2}\,h>g_0=0.64$.

Case II. $\qquad \Omega=0,\ \rho_0=0.$

In this case, in the absence of rotation and outside fluid, the dispersion relation reduces to

$$4\pi G\rho\sqrt{k^2+l^2}\,h\tanh(\sqrt{k^2+l^2}\,h)\left\{1-\frac{1}{h(k^2+l^2)^{\frac{1}{2}}\{1+\tanh\sqrt{k^2+l^2}\,h\}}\right.$$

$$\left.+\frac{\mu H^2k^2}{4\pi G\rho^2 h\sqrt{k^2+l^2}\tanh(\sqrt{k^2+l^2}\,h)}\right\}=\sigma^2. \qquad \ldots \quad (39)$$

This result has been obtained by Oganesyan (1961).

Case III. $\qquad \rho_0=0,\quad H=0.$

The dispersion relation reduces to

$$\left\{\frac{4\pi G\rho^2}{\sqrt{k^2+l^2}\{1+\tanh (k^2+l^2)^{\frac{1}{2}}h)\}}-4\pi G\rho^2 h\right]$$

$$\times\sqrt{k^2+l^2}\tan\left(\frac{\sqrt{k^2+l^2}\,\sigma}{\sqrt{4\Omega^2-\sigma^2}}\right)h-\rho\sigma(4\Omega^2-\sigma^2)^{\frac{1}{2}}=0. \qquad \ldots (40)$$

By investigating the nature of σ in the dispersion relation (40), it is possible to show that there [illegible]bility if

$$\frac{\pi G\rho}{\Omega^2}\left[\frac{1}{h\sqrt{k^2+l^2}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}}-1\right]h^2(k^2+l^2)<1,$$

and instability if the left hand side of the inequality is greater than unity (see the Appendix). Thus the critical value of $h\sqrt{k^2+l^2}$ at which instability sets in is given by the equation

$$\frac{\pi G\rho}{\Omega^2}\left[\frac{1}{h\sqrt{k^2+l^2}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}}-1\right]h^2(k^2+l^2)=1. \qquad \ldots (41)$$

We also note that the dispersion relation (40) reduces to (41) as $\sigma\to 0$, i.e. the principle of exchange of stability holds good (cf. Chandrasekhar 1952).

We have computed the value of

$$\left[\frac{1}{h\sqrt{k^2+l^2}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}}-1\right]h^2(k^2+l^2)$$

for different values of $h\sqrt{k^2+l^2}$. We find that this has the maximum value 0.143 at $h(k^2+l^2)^{\frac{1}{2}}=0.30$. Thus the left hand side of (41) is always less than unity if

$$\frac{\pi G\rho}{\Omega^2}<\frac{1}{0.143} \qquad \ldots (42)$$

Hence if (42) is satisfied there is always stability. With $\Omega\sim 10^{-15}$ per sec. (42) suggests that for a density $\rho<10^{-21}$ gm/cc the system is stable. The values of Ω and ρ noted above are obtained in our galaxy (Chandrasekhar, 1955).

It may be noted that the result (42) is independent of the thickness h of the fluid layer, and thus valid for all values of h.

Case IV. $\rho_0=0$.

In this case, when $\sigma\to 0$, the dispersion relation reduces to

$$4\pi G\rho\sqrt{k^2+l^2}h\tanh(\sqrt{k^2+l^2}h)\times\left[1-\frac{1}{\sqrt{k^2+l^2}h\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}}\right]$$

$$+\frac{\eta k^2h^2n^2}{\sqrt{k^2+l^2}h\tanh(h\sqrt{k^2+l^2})}=0, \qquad \ldots (43)$$

where

$$\eta=\frac{H}{4\pi\sqrt{G\rho h}}. \qquad \ldots (44)$$

The equation (43) is the same as that obtained by Oganesyan (1961) in the absence of rotation for the critical values of the wave numbers at which instability sets in. On examining the dispersion relation (37) for values of $\sqrt{k^2+l^2}\,h$ and kh which are in a small neighbourhood of (y_1, y_2) where $\sqrt{k^2+l^2}\,h = y_1$ and $kh = y_2$ are roots of (43), it is clear that we can find $\sqrt{k^2+l^2}\,h$ and kh for which thre is instability whatever be the values of ρ and Ω. Thus the magnetic field has a destabilizing effect.

APPENDIX

We shall briefly prove that the principle of the exchange of stability holds in case III, i.e. $\sigma = 0$ gives the critical wave-lengths at which instability sets in.

Let

$$\frac{\sigma}{\sqrt{4\Omega^2-\sigma^2}} = X.$$

Then the dispersion relation in case III reduces to

$$\left[\frac{4\pi G\rho^2}{(k^2+l^2)^{\frac{1}{2}}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}} - 4\pi G\rho^2 h\right] \times (1+X^2)\sqrt{k^2+l^2}$$

$$\times \sin(\sqrt{k^2+l^2}\,hX) = 4\Omega^2 X\rho\cos(\sqrt{k^2+l^2}\,hX). \qquad \ldots \text{(A-1)}$$

Let $|X| = R$ be large so that R is the n-th root of $\cos(\sqrt{k^2+l^2}\,hX) = 0$ where n is large. We note that for $|X| = R$,

$|\cos(\sqrt{k^2+l^2}\,hX)|$ and $|\sin(\sqrt{k^2+l^2}\,hX)|$ are of the same order of magnitude, except near $X = \pm R$, where $\left|\frac{\sin(\sqrt{k^2+l^2}\,hX)}{\cos(\sqrt{k^2+l^2}\,hX)}\right|$ tends to an indefinitely large quantity as $X \to \pm R$. Hence by Rouch's theorem the equation (A-1) has the same number of roots within $|X| = R$ as the equation

$$(1+X^2)\sqrt{k^2+l^2}\sin(\sqrt{k^2+l^2}\,hX) = 0. \qquad \ldots \text{(A-2)}$$

Now, $X = 0$ is a root of both (A-1) and (A-2). Writing (A-1) as

$$\left[\frac{4\pi G\rho^2}{(k^2+l^2)^{\frac{1}{2}}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}} - 4\pi G\rho^2 h\right]$$

$$\times \frac{\sqrt{k^2+l^2}}{4\Omega^2\rho}\tan(\sqrt{k^2+l^2}\,hX) = \frac{X}{1+X^2}, \qquad \ldots \text{(A-3)}$$

we draw the graphs of the right hand side as well as left hand side of (A-3) against real positive values of X. (The equation (A-1) or (A-3) has equal and opposite roots in pairs). It is clear that corresponding to the n roots of $\cos(\sqrt{k^2+l^2}\,hX) = 0$ within the circle $|X| = R$ there are n vertical asymptotes of the graph of the left hand side of (A-3) and there will be $n-1$ intersections of the graph of the left hand side with that of the right hand side between $X = \beta_1$ and $X = \beta_n$ where β_1 and β_n are the first and the n-th roots of

$$\cos\left(\sqrt{k^2+l^2}\, hX\right) = 0. \quad \left(\beta_1 = \frac{\pi}{2h\sqrt{k^2+l^2}}\right).$$

Thus $X = 0$ and $n-1$ other roots between $X = \beta_1$ and $X = \beta_n$ for (A-3) are located corresponding to the n roots of $\sin\left(\sqrt{k^2+l^2}\, hX\right) = 0$ between $X = 0$ and $X = \beta_n$. But the roots of (A-3) that correspond to the roots $X = \pm i$ of (A-2) are still to be accounted for.

Considering the growth rates of the left hand side and the right hand side of (A-3) between $X = 0$ and $X = \beta_1$ it can be shown that there is one root between $X = 0$ and $X = \beta_1$ when

$$\alpha = \frac{\pi G\rho h^2(k^2+l^2)}{\Omega^2}\left[\frac{1}{h(k^2+l^2)^{\frac{1}{2}}\{1+\tanh(k^2+l^2)^{\frac{1}{2}}h\}} - 1\right]$$

lie between 0 and 1.

If $\alpha > 1$, then there is no root (real) between $X = 0$ and $X = \beta_1$. In this case only possibility is that there will be a complex or a purely imaginary root $X = iX_1$ (with real X_1), such that

$$1 > X_1^2.$$

If $\alpha < 0$, there is a root $X = iX_1$ such that $X_1 > 1$.

From the relation $\dfrac{\sigma}{\sqrt{4\Omega^2-\sigma^2}} = X$, it is clear that when $\alpha > 1$, σ is either imaginary or complex. From (A-1), it is clear that the dispersion relation is even in X and σ. Hence equal and opposite roots σ occur in pairs. Thus imaginary or complex σ indicates instability.

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